On the canonical threefolds with strictly nef anticanonical divisors

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Abstract

The purpose of this article is to give a criterion for canonical 3-folds to be a $\mathbb{Q}$-Fano 3-folds.

0   Introduction

We are working in the category of projective varieties defined over the field of complex numbers.

We study a numerical criterion for canonical 3-folds to be $\mathbb{Q}$-Fano 3-folds. A $\mathbb{Q}$-Fano 3-fold is defined in this paper to be a normal projective variety of dimension 3 having only canonical singularities with its anticanonical divisor being ample.

First of all, we state the following famous criterion for ampleness of divisors, so-called Kleiman's criterion.

Theorem 0.1. **[15]** *Let X be a projective variety and D a Cartier divisor. Then D is ample if and only if $D \cdot z > 0$ for all $z \in \overline{NE}(X) \backslash \{0\}$.*

Next we introduce the notion, *strictly nef*.

Definition 0.2. Let X be a normal projective variety and D a $\mathbb{Q}$-Cartier Weil divisor. D is *strictly nef*, if $D \cdot C > 0$ for all irreducible curves C.

We are interested in how far strictly nef divisors are from being ample. Mumford gave an example of a strictly nef non-ample divisor on a surface (see [7], p56). In words in Theorem 0.1, $D \cdot z > 0$ for all $z \in NE(X) \backslash \{0\}$ does not imply that D is ample. But in the case $D = -K_X$, we have the following results by H. Maeda, F. Serrano and K. Matsuki.

Theorem 0.3. *Let X be a $\mathbb{Q}$-Gorenstein normal projective variety with strictly nef $-K_X$. Assume that one of the following conditions holds.*

(i) [19] *X is smooth and* $\dim X = 2$.

(ii) [25] *X is smooth and* $\dim X = 3$.

(iii) [20] *X is a canonical 3-fold and* $\kappa(-K_X) \geq 0$.

Then $-K_X$ *is ample.*

Furthermore in more general aspect, we have

Theorem 0.4. [25] *Let X be an irreducible reduced projective Gorenstein surface and L a strictly nef Cartier divisor. Then* $mL + K_X$ *is ample for a sufficiently large integer m.*

Our purpose in this paper is to prove the following theorem. We will prove it with the aid of various results including Theorem 0.3 and 0.4.

Main Theorem (=Theorem 3.8). *Let X be a canonical 3-fold with strictly nef* $-K_X$. *Then* $-K_X$ *is ample.*

The proof rests mainly on sophisticated techniques of three dimensional birational geometry developed in the last two decades, essentially by the Japanese School. We now give an outline of this article. Let X be as in the Main Theorem. Let $\alpha : X \to \mathrm{Alb}(X)$ be the Albanese morphism and $\mathrm{alb}(X)$ denotes $\dim \alpha(X)$. We will show in §1 that the case $\mathrm{alb}(X) = 1$ never occurs and in §2 that the case $\mathrm{alb}(X) = 2$ never occurs by the extremal ray theory. Since $\kappa(X) = -\infty$, we can check $\mathrm{alb}(X) \neq 3$ easily (Remark 3.7). After we know them, it is enough to show the ampleness of $-K_X$ with the assumption $q(X) = 0$ in order to prove the Main Theorem. It will be done in §3.

Notation and Convention.
(i) We work over the field of complex numbers.
(ii) A terminal (resp. canonical) 3-fold means a normal projective 3-dimensional variety with at most terminal (resp. canonical) singularities.

(iii) A surjective morphism between normal projective varieties is an algebraic fiber space if it has connected fibers.

(iv) An algebraic fiber space $\varphi : X \to Y$ between normal projective varieties is an extremal contraction if $-K_X$ is φ-ample and if the Picard numbers satisfy $\rho(X) = \rho(Y) + 1$.

(v) A birational morphism between normal projective varieties is small if it is isomophic in codimension 1.

(vi) We always denote numerical equivalence by $\equiv$.

(vii) Let X be as in the Main Theorem. We always denote the Albanese morphism of X by α in this paper. Define $\mathrm{alb}(X) := \dim \alpha(X)$.

In the whole paper we will freely use the results from classification theory and Mori theory. Refer to [13] and [18].

1 Proof of $\mathrm{alb}(X) \neq 1$

A normal variety X is said to have only *terminal singularities* (resp. *canonical singularities*) if the following conditions are satisfied:

(i) The canonical divisor K_X is $\mathbb{Q}$-Cartier, i.e. mK_X is a Cartier divisor for some positive integer m.

(ii) For a resolution of singularities $f : Y \to X$ we have

$$K_Y = f^* K_X + \sum a_i E_i$$

with $a_i > 0$ (resp. $a_i \geq 0$) for all i, where the E_i's run through all the prime divisors on Y which are exceptional with respect to f. In paticular if X is a surface, X has only terminal singularities (resp. canonical singularities) is equivalent to that X is nonsingular (resp. has only Du Val singularities, i.e. rational double points).

Let X be a normal projective variety with at most canonical singularities. Then X has rational singularities ([4]) so the Albanese map of X is a morphism (cf. Lemma 8.1 in [10]).

Assume in addition that $\dim X = 3$. Because a general hyperplane section of X has at most Du Val singularities, the locus of non-cDV (compound Du Val) points is a finite set. See also Corollary 5.40 in [18].

We enjoy the benefit of the following result repeatedly in this paper.

Proposition 1.1. [14] *Let X be a $\mathbb{Q}$-factorial terminal 3-fold such that $-K_X$ is nef, but not big. Then $c_1(X)c_2(X) \geq 0$.*

Remark 1.2. Taking a $\mathbb{Q}$-factorialization allows us to remove the assumption $\mathbb{Q}$-factoriality in Proposition 1.1. For the definition of $\mathbb{Q}$-factorialization, see Remark 1.12. By [11], there exists a nonnegative rational number A such that $\chi(\mathcal{O}_X) = \frac{c_1 c_2}{24} + A$, where $A = 0$ if and only if X is Gorenstein. Thus assume in addition that $\chi(\mathcal{O}_X) = 0$ in the proposition above. Then X is Gorenstein.

Lemma 1.3. [25] *Let Y be a smooth projective variety, C a smooth projective curve of positive genus, and $g : Y \to C$ a morphism with connected fibers. We denote by F_t the fiber over $t \in C$. Let D be a divisor on Y and pick a paticular fiber F_0, $0 \in C$. If $H^0(F_0, D|_{F_0}) \neq 0$, and $H^1(Y, D + F_t - F_0) = 0$ for all $t \in C$, then D is algebraically equivalent to an effective divisor.*

Using the idea of [25], we can show the following proposition rather easily. The key fact of the proof is that $\mathbb{Q}$-factorial Gorenstein terminal 3-folds are factorial (Lemma 5.1 in [12]).

Proposition 1.4. *Let X be a $\mathbb{Q}$-factorial terminal 3-fold with strictly nef $-K_X$. Then $q(X) = 1$ never occurs.*

Proof. Assume that $q(X) = 1$. Then α is an algebraic fiber space and $C := \mathrm{Alb}(X)$ is an elliptic curve. Let F be a general fiber of α. Then F is a smooth projective surface, so $-K_F = -K_X|_F$ is ample (Theorem 0.3). Hence $R^i \alpha_* \mathcal{O}_X = 0$ for $i > 0$ by Kawamata-Viehweg Vanishing Theorem (cf. [13], Theorem 1.2.5) and so $h^i(\mathcal{O}_X) = h^i(\mathcal{O}_C)$ for all i. Then applying Remark 1.2, we know that X is a $\mathbb{Q}$-factorial Gorenstein terminal 3-fold.

Let $f : Y \to X$ be a resolution and put $g := \alpha \circ f$.

<u>Claim.</u> $-mK_X \equiv E$ for some effective divisor E and some $m > 0$.

Proof of the claim. Set X_t (resp. Y_t) the fiber of α (resp. g) over $t \in C$. For a sufficiently large integer m,

$$h^0(F, -mf^*K_X|_F) = h^0(F, -mK_F) \neq 0 \tag{1}$$

by the ampleness of $-K_F$. Fix this m. We may assume $h^0(-mf^*K_X + F - Y_t) = h^0(-mK_X + F - X_t) = 0$. Let S be a sufficiently ample line bundle on X. We consider the following exact sequence.

$$\cdots \to H^1((m+1)K_X - F + X_t - S) \to H^1((m+1)K_X - F + X_t)$$
$$\to H^1((m+1)K_X - F + X_t|_S) \to \cdots$$

Since $(-K_X)^2 \not\equiv 0$, $-(m+1)K_X + F - X_t|_S$ is nef and big. Thus

$$h^1((m+1)K_X - F + X_t|_S) = h^1(-(m+1)K_X + F - X_t|_S + K_S) = 0.$$

Therefore looking at the exact sequence above,

$$h^2(-mf^*K_X + F - Y_t) = h^2(-mK_X + F - X_t) = h^1((m+1)K_X - F + X_t) = 0.$$

Thus

$$-h^1(-mf^*K_X + F - Y_t) = \chi(-mK_X + F - X_t) = (2m+1)\chi(\mathcal{O}_X) = 0 \tag{2}$$

by Riemann-Roch Theorem (cf. [24]). Then we can apply Lemma 1.3 because of (1) and (2). Thus we know that $-mf^*K_X$ is algebraically equivalent to an effective divisor. The algebraically equivalence of cycles is preserved by proper push-forward (see [5], Proposition 10.3). Then we obtain the claim.

We can write $-K_X \equiv \sum a_i Z_i$ for some $a_i \in \mathbb{Q}_{>0}$ and some prime divisors Z_i by the claim. Since $-K_X$ is nef but not big by $q(X) \neq 0$, $(-K_X)^3 = 0$ and so $(-K_X)^2 Z_i = 0$ for all i. Because Z_i is Cartier by Lemma 5.1 in [12], Z_i is a Gorenstein surface. Hence $K_{Z_i} - nK_X|_{Z_i}$ is ample for $n \gg 0$ (Theorem 0.4), then

$$0 < -K_X|_{Z_i}(K_{Z_i} - nK_X|_{Z_i}) = -K_X Z_i(K_X + Z_i - nK_X) = -K_X Z_i{}^2.$$

The absurdity comes from

$$0 = (-K_X)^2 Z_1 = (-K_X)(\sum a_i Z_i)Z_1 = a_1(-K_X)Z_1{}^2 + (-K_X)\sum_{i \neq 1} a_i Z_1 Z_i > 0.$$

$\square$

Lemma 1.5. *Let X be a canonical 3-fold with strictly nef $-K_X$. Assume in addition that $\mathrm{alb}(X) = 1$. Then $q(X) = 1$.*

Proof. Note that α is an algebraic fiber space in the case $\mathrm{alb}(X) = 1$ when we regard α as a morphism from X to $\alpha(X)$. Assume $q(X) > 1$. Denote a terminalization of X by $f : Y \to X$ (see Remark 1.10). Let F be a general fiber of α. Then $F' := f^*(F)$ is smooth and connected, so F is an irreducible reduced Gorenstein surface. Therefore $-K_F = -K_X|_F$ is ample by Theorem 0.4 and we obtain $R^i \alpha_* \mathcal{O}_X = 0$ for all $i > 0$ by relative Kawamata-Viehweg Vanishing Theorem. Then $h^i(\mathcal{O}_Y) = h^i(\mathcal{O}_X) = h^i(\mathcal{O}_{\alpha(X)})$ for all i, and so $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_{\alpha(X)}) < 0$. This lies in contradiction with Proposition 1.1. $\square$

The following definition is in [22]. It is useful, for example, when we investigate a $\mathbb{Q}$-factorial terminal 3-fold with nef anticanonical divisor.

Definition 1.6. Let X be a normal projective variety and D a $\mathbb{Q}$-Cartier Weil divisor on X. Then D is called *almost nef*, if there are at most finitely many rational curves C_i, $1 \le i \le r$, such that $D \cdot C \ge 0$ for all curves $C \ne C_i$.

Proposition 1.7. [22] *Let X be a $\mathbb{Q}$-factorial terminal 3-fold with almost nef $-K_X$. Assume that one of the following conditions holds.*

   (i) There is an extremal contraction $\varphi : X \to C$ to the elliptic curve C.

   (ii) $q(X) = 1$ and there exists an extremal contraction $\varphi : X \to W$ to the surface W .

   (iii) $\mathrm{alb}(X) = 2$ and there exists an extremal contraction $\varphi : X \to W$ to the surface W .

Then the Albanese morphism $\alpha : X \to \mathrm{Alb}(X)$ is smooth and $-K_X$ is nef.

We will use Proposition 1.7 under the condition (iii) only.

Proposition 1.8. [22] *Let X be a $\mathbb{Q}$-factorial terminal 3-fold with almost nef $-K_X$.*

   (i) Let $\varphi : X \to W$ be a divisorial contraction or a flip. Then $-K_W$ is almost nef.

   (ii) Let $\varphi : X \to W$ be an extremal contraction to the surface W . Then $-(4K_W + \Delta)$ is almost nef (see the definition of Δ in [22], which is an effective divisor on W).

Remark 1.9. Let X be a $\mathbb{Q}$-factorial terminal 3-fold with nef $-K_X$ and $\varphi : X \to W$ a divisorial contraction. This does not imply that $-K_W$ is *nef* (see Proposition 3.3 in [3]). As in [22], this is the reason why they introduce the notion *almost nef*.

Remark 1.10. Let X be a 3-fold with canonical singularities. According to [23], there exists a partial resolution $f : Y \to X$ such that (i) Y has terminal singularities, and (ii) f is crepant and projective. We call f and Y terminalization of X.

We have more information on terminalization as follows. We denote the number of crepant divisors over a canonical 3-fold X by $e(X)$.

Proposition 1.11. [12], [17] *Let X be a normal projective 3-dimensional variety having at most $\mathbb{Q}$-factorial canonical singularities. Assume that $e(X) > 0$. Then there exist a normal projective variety X_1 having at most $\mathbb{Q}$-factorial canonical singularities and a projective birational morphism $g : X_1 \to X$ such that the following conditions are satisfied:*

(i) g is crepant;

(ii) The exceptional locus of g is a prime divisor;

(iii) $\rho(X_1/X) = 1$ and $e(X_1) = e(X) - 1$.

Using Proposition 1.11 iterately, for a given $\mathbb{Q}$-factorial canonical 3-fold X, we obtain its terminalization because $e(X) < \infty$.

Remark 1.12. Recall that a $\mathbb{Q}$-factorialization of a canonical 3-fold X means a birational projective morphism $f : Y \to X$ such that (i) Y has at most $\mathbb{Q}$-factorial canonical singularities, and (ii) f is small. Yujiro Kawamata proved in [12] the existence of $\mathbb{Q}$-factorialization of canonical 3-folds. Note that if $f(C)$ is a point for an irreducible curve C on Y, then $C = \mathbb{P}^1$ because $R^i f_* \mathcal{O}_Y = 0$ for $i > 0$.

The following proposition is the main result of this section.

Proposition 1.13. *Let X be a canonical 3-fold with strictly nef $-K_X$. Then $\mathrm{alb}(X) \neq 1$.*

Proof. Assume that $\mathrm{alb}(X) = 1$. Then we know that $q(X) = 1$ because of Lemma 1.5 and $\mathrm{Alb}(X)$ is an elliptic curve. Let $g_0 : Y_0 \to X$ be a $\mathbb{Q}$-factorialization. Then there exists a sequence

$$Y := Y_n \to Y_{n-1} \to \cdots \to Y_1 \to Y_0 \to X.$$

such that for $i \geq 1$ each morphism $g_i : Y_i \to Y_{i-1}$ is constructed by Proposition 1.11 and $Y := Y_n$ has at most $\mathbb{Q}$-factorial terminal singulalities (if $e(Y_0) = 0$, $n = 0$). Put $f := g_0 \circ g_1 \circ \cdots \circ g_n$. Since K_Y is not nef, there exists an extremal contraction $\varphi : Y \to W$. Since $h^i(\mathcal{O}_Y) = h^i(\mathcal{O}_{\mathrm{Alb}(X)})$ for $i \geq 0$ as in the proof of Lemma 1.5, we obtain that Y is Gorenstein (Remark 1.2). Furthermore we know that $\dim W \geq 1$ easily. Let $\beta : W \to \mathrm{Alb}(W)(= \mathrm{Alb}(X))$ be the Albanese morphism.

Case: $\dim W = 1$
In this case, W is an elliptic curve and so β is an isomorphism. Proposition 1.4 says that X is not $\mathbb{Q}$-factorial terminal, so there exists an irreducible

curve C such that $f(C)$ is a point. Then $K_Y \cdot C = 0$ and so $\varphi(C)$ is a curve. This lies in contradiction with $\alpha \circ f = \varphi$.

Case: $\dim W = 2$

When $\kappa(W) \geq 0$, $K_W \cdot H \geq 0$ for some ample divisor H. But because $-(4K_W + \Delta)$ is almost nef (Proposition 1.8), $-K_W \cdot H \geq 0$. Therefore we know $K_W \equiv 0$. Combining this with $h^i(\mathcal{O}_W) = h^i(\mathcal{O}_Y) = h^i(\mathcal{O}_{\mathrm{Alb}(X)})$ for all i, we know that W is a hyperelliptic surface. If X has only terminal singularities, f is just a $\mathbb{Q}$-factorialization so it is small. X is not $\mathbb{Q}$-factorial terminal by Proposition 1.4, then there exists a rational curve C on Y such that $K_Y \cdot C = 0$, and so $\varphi(C)$ is a rational curve on a hyperelliptic surface W. This is absurd. Thus $e(X) > 0$, that is to say, X is not terminal and there exists a morphism $g_n : Y \to Y_{n-1}$ such that g_n is constructed by Proposition 1.11 and f factors through g_n. Put D the exceptional divisor of g_n. If $\dim g_n(D) = 0$, then D is uniruled by the Subadjunction Lemma (Lemma 5.1.9 ! of [13]). But this lies in contradiction with $\varphi(D) = W$. If $\dim g_n(D) = 1$, then there exists an irreducible curve C on Y such that $g_n(C)$ is a point. Since $\dim g_n^{-1}(g_n(C)) = 1$, C is a smooth rational curve. But this derives contradiction as above.

Assume that $\kappa(W) = -\infty$ below. We obtain that $-K_Y \equiv \sum a_i Z_i$ for some $a_i \in \mathbb{Q}_{>0}$ and some prime divisors Z_i as in the proof of Proposition 1.4. Since $(-K_Y)^3 = 0$, $(-K_Y)^2 \cdot Z_i = 0$ and there exists an irreducible curve $C_i' \subset Z_i$ such that $f(C_i')$ is a point for all i. In fact, if not, $-K_Y|_{Z_{i_0}}$ is strictly nef for some i_0. Moreover we know by Theorem 0.4 that

$$0 < -K_Y|_{Z_{i_0}}(K_{Z_{i_0}} - nK_Y|_{Z_{i_0}}) = -K_Y Z_{i_0}(K_Y + Z_{i_0} - nK_Y) = -K_Y Z_{i_0}^2.$$

for $n \gg 0$. Then we can derive a contradiction as follows:

$$0 = (-K_Y)^2 Z_{i_0} = (-K_Y)(\sum a_i Z_i) Z_{i_0} = a_{i_0}(-K_Y)(Z_{i_0})^2 + (-K_Y) \sum_{i \neq i_0} a_i Z_{i_0} Z_i > 0.$$

Let $\gamma : Y \to \mathrm{Alb}(Y) = (\mathrm{Alb}(X))$ be the Albanese morphism. Note that $\alpha \circ f = \gamma = \beta \circ \varphi$.

$$
\begin{array}{ccc}
X & \xleftarrow{\;f\;} & Y \\
{\scriptstyle\alpha}\downarrow & \swarrow{\scriptstyle\gamma} & \downarrow{\scriptstyle\varphi} \\
\mathrm{Alb}(X) & \xleftarrow{\;\beta\;} & W
\end{array}
$$

<u>Claim.</u> $\varphi(Z_{i_0}) = W$ for some i_0.

Proof of the claim. If $\gamma(Z_i)$ is a point for all i, then there exists an irreducible curve C on Y such that $f(C)$ is 1-dimensional and C does not meet any Z_i. This implies

$$0 < -K_Y \cdot C = \sum a_i Z_i \cdot C = 0$$

which is contradiction. Therefore we know that $\gamma(Z_{i_0}) = \mathrm{Alb}(X)$ for some i_0, then $\varphi(Z_{i_0}) = W$. In fact, assume to the contrary that $\varphi(Z_{i_0})$ is a curve. Now that $\varphi(C'_{i_0})$ is a curve, $\mathrm{Alb}(X) = \gamma(Z_{i_0}) = \beta \circ \varphi(Z_{i_0}) = \beta \circ \varphi(C'_{i_0})$. But this is absurd because $\beta \circ \varphi(C'_{i_0})$ is a point.

Using the claim above, we prove the following.

<u>Claim.</u> Assume that $-(4K_W + \Delta) \cdot \tilde{C} = 0$ for an irreducible curve $\tilde{C}$ on W. Then $\tilde{C}$ is a fiber of β.

Proof of the claim. If $f(C)$ is a curve for all curves $C \subset \varphi^{-1}(\tilde{C})$, then $Z_i \not\subset \varphi^{-1}(\tilde{C})$ for all i by $C'_i \subset Z_i$. From the well known formula $(K_Y)^2 \cdot \varphi^{-1}(\tilde{C}) = -(4K_W + \Delta) \cdot \tilde{C}$ (see the proof of Lemma 1.6 in [22]), we get

$$-K_Y \cdot \left(\sum a_i Z_i \right) \cdot \varphi^{-1}(\tilde{C}) = 0.$$

Then $\dim Z_i \cap \varphi^{-1}(\tilde{C}) \leq 0$ for all i. This lies in contradiction with $\varphi(Z_{i_0}) = W$. Then we know that there exists a curve $C \subset \varphi^{-1}(\tilde{C})$ such that $f(C)$ is a point, and so $\tilde{C} = \varphi(C)$ is a fiber of β.

It follows from Proposition 1.7 in [22] that W is a $\mathbb{P}^1$-bundle over $\mathrm{Alb}(X)$ with nef $-K_W$ and almost nef $-(4K_W + \Delta)$. Then we can write $W = \mathbb{P}^1(V)$ for a normalized sheaf V on C, i.e. $H^0(V) \neq 0$, but $H^0(V \otimes \mathcal{L}) = 0$ for all line bundles $\mathcal{L}$ with $\deg \mathcal{L} < 0$ (see [7] about the treatment and the terminology of geometrically ruled surfaces around here). Set $e := -c_1(V)$. We separate into these two cases: $e = 0$ or -1. Let F be a fiber of β and C_0 a canonical section satisfying $C_0^2 = -e$.

In the case $e = -1$, there exists an irreducible curve C_1 on W such that C_1 is numerically proportional to $2C_0 - F$. Since C_1 is nef,

$$0 \leq -(4K_W + \Delta) \cdot C_1 = -\Delta \cdot C_1 \leq 0.$$

Thus

$$-(4K_W + \Delta) \cdot C_1 = 0.$$

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This contradicts the claim above.

In the case $e = 0$, C_0 is nef. Then we obtain

$$0 \leq -(4K_W + \Delta) \cdot C_0 = -\Delta \cdot C_0 \leq 0.$$

This also yields contradiction.

Case: $\dim W = 3$

Since Y is Gorenstein, φ is a divisorial contraction (cf. [1]). Let E be the exceptional divisor. If $\varphi(E)$ is a point, then we have the following equation

$$K_Y = \varphi^* K_W + aE$$

for some $a \in \mathbb{Q}_{>0}$. Thus $-K_W$ is nef. Furthermore looking at the description of the normal bundle of E in Y, $\mathcal{N}_{E/Y}$, in [2], we get

$$(-K_W)^3 = (-K_Y)^3 + a^3 E^3 \geq a^3 E^3 > 0$$

(note that $(\varphi^* K_W + aE)^3 = K_W{}^3 + a^3 E^3$ since $\varphi(E)$ is a point). Hence $q(W) = 0$ by Kawamata-Viehweg Vanishing Theorem. This derives contradiction.

Hence we shall assume that $\varphi(E)$ is a curve from now on. We know that $C_0 := \varphi(E)$ is locally a complete intersection and φ is just the blow-up of W along C_0 (cf. [2]). Then W being Cohen-Macaulay implies $E = \mathbb{P}(\mathcal{N}^*_{C_0/W})$, where $\mathcal{N}_{C_0/W}$ denotes the normal bundle of C_0 in W. Let $\nu : C_0' \to C_0$ be the normalization, V the normalized sheaf of $\nu^*(\mathcal{N}^*_{C_0/W})$, i.e. $H^0(V) \neq 0$, but $H^0(V \otimes \mathcal{L}) = 0$ for all line bundles $\mathcal{L}$ with $\deg \mathcal{L} < 0$. Furthermore let $\psi : \mathbb{P}(V)(=: E') \to C_0'$ be the projection and F its fiber. The tautological line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$ has a canonical section C_1 satisfying $C_1{}^2 = c_1(V)(=: -e)$. Put g the genus of C_0'.

$$X \xleftarrow{\ f\ } Y \supset E \xleftarrow{\ h\ } E'$$
$$\downarrow{\varphi} \qquad \downarrow \qquad\quad \downarrow{\psi}$$
$$W \supset C_0 \xleftarrow{\ \nu\ } C_0'$$

We have

$$K_Y = \varphi^* K_W + E, \tag{1}$$

and since $\mathcal{N}_{E/Y} = \mathcal{O}_E(-1)$

$$h^*(\mathcal{N}_{E/Y}) \equiv -C_1 + aF \qquad (a \in \mathbb{Z}) \tag{2}$$

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where $h : E' \to E$ is the base change of ν by φ. We can write $V = \nu^*(\mathcal{N}^*_{C_0/W}) \otimes \mathcal{M}$ for some $\mathcal{M} \in \mathrm{Pic}(C_0')$. Then note that $a = \deg \mathcal{M}$. Moreover

$$h^*(-K_Y|_E) \equiv C_1 + bF \qquad (b \in \mathbb{Z}) \tag{3}$$

from $h^*(K_Y|_E) \cdot F = h^*(\mathcal{N}_{E/Y}) \cdot F = -1$. Combining (1),(2) and (3), we get

$$\begin{aligned} -K_W \cdot C_0 = -K_W \cdot \varphi_* h_* C_1 &= (-K_Y + E) \cdot h_* C_1 \\ &= h^*(-K_Y + E|_E) \cdot C_1 = a + b, \end{aligned} \tag{4}$$

$$(-K_Y)^2 E = h^*(-K_Y|_E)^2 = -e + 2b, \tag{5}$$

$$E^2(-K_Y) = h^*(\mathcal{N}_{E/Y}) \cdot h^*(-K_Y|_E) = e + a - b, \tag{6}$$

$$(-K_W)^3 = 3K_Y{}^2 E - 3K_Y E^2 + E^3 = -e + a + 3b, \tag{7}$$

and

$$h^*(K_E) = h^*(K_Y + E|_E) \equiv -2C_1 + (a - b)F.$$

On the other hand, since $\omega_{E'}$ is a subsheaf of $h^*\omega_E$, the last equation implies

$$K_{E'} + cF \equiv -2C_1 + (a - b)F \qquad (c \geq 0).$$

Squaring yields

$$2(1 - g) - c = -e - a + b. \tag{8}$$

Note that $-K_W \cdot C \geq 0$ for any curves C in W, $C \neq C_0$, and so if $a + b \geq 0$, $-K_W$ is nef by (4). Furthermore we know $-e + 2b \geq 0$ by (5).

If $e + a - b > 0$, we get

$$-K_W \cdot C_0 = a + b \geq e + a - b > 0$$

and

$$(-K_W)^3 = -e + a + 3b \geq e + a - b > 0.$$

Hence $-K_W$ is nef and big, so this contradicts $q(W) = q(X) \neq 0$. Consequently we know

$$e + a - b \leq 0.$$

So we obtain from (8) that $2(1-g) \geq c$. Since $c \geq 0$, it is enough to consider the following four cases:

Case 1: $\quad g = 0$, $c = 0$, Case 2: $\quad g = 0$, $c = 1$,

Case 3: $\quad g = 0$, $c = 2$, and Case 4: $\quad g = 1$, $c = 0$.

Before we consider the four cases above, we prove the following claim.

<u>Claim.</u> If $g = 0$, then $-e + 2b > 0$.

Proof of the claim. Assume that $-e + 2b = 0$. We have

$$0 \leq h^*(-K_Y|_E) \cdot C_1 = -e + b = -\frac{1}{2}e.$$

Thus $e = 0$ by $g = 0$ and $K_Y \cdot h_* C_1 = 0$. Therefore E' is $\mathbb{P}^1 \times \mathbb{P}^1$ and $f \circ h(C_1')$ is a point for any curves $C_1' \in |C_1|$ where $|C_1|$ is the complete linear system on E' defined by C_1. Using (8), we obtain that $a = -2$ in Case 1, $a = -1$ in Case 2 and $a = 0$ in Case 3.

$$X \xleftarrow{g_0} Y_0 \xleftarrow{g_1} \cdots \xleftarrow{g_{n-1}} Y_{n-1} \xleftarrow{g_n} Y_n = Y \supset E \xleftarrow{h} E' = \mathbb{P}^1 \times \mathbb{P}^1$$

with vertical maps φ and ψ, and

$$W \supset C_0 \xleftarrow{\nu} C_0' = \mathbb{P}^1$$

Let D_i be the exceptional divisor of g_i for $i \geq 1$. Then there exists $m \geq 1$ such that $D_m = g_{m+1} \circ \cdots \circ g_n(E)$ and $\dim g_m(D_m) = 1$. We denote the image of C_1 on Y_m by C and the strict transform of D_i on Y by $\tilde{D}_i$. We know that $g_m(C)$ is a point and $\dim g_m^{-1}(g_m(C)) = 1$. Thus $C = \mathbb{P}^1$ by $R^1 g_{m*} \mathcal{O}_{Y_m} = 0$. Put $g_n^* \cdots g_{m+1}^* D_m = E + \sum_{i=m+1}^n a_i \tilde{D}_i$ for some $a_i \in \mathbb{Q}_{\geq 0}$. Then by (2),

$$D_m \cdot C = (E + \sum a_i \tilde{D}_i) \cdot h(C_1) \geq E \cdot h(C_1) = a.$$

On the other hand, let H be a general hyperplane section of Y_{m-1}. Then the induced morphism $g_{m,H} : H' := g_m^* H \to H$ is a crepant partial resolution. Now that C is a smooth rational curve,

$$D_m \cdot C = D_m|_{H'} \cdot C = C^2 = -2.$$

Therefore we obtain $a = -2$. Moreover we get $c = 0$ by (8), so $C_0 = \mathbb{P}^1$, $E = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{N}_{C_0/W} = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ by the definition of a and

c. Then $K_Y{}^2 \equiv 0$ by Proposition 3.11 in [3]. This yields a contradiction because $K_X{}^2 F_\alpha > 0$ for a general fiber F_α of α.

Now we shall investigate each case.

Case 1: $g = 0$ and $c = 0$.
In this case, we know that C_0 is normal by the definition of c, and so ν and h are isomorphisms. From (8), we obtain

$$-e - a + b = 2. \tag{9}$$

If $-e + 2b \geq 2$, then

$$(-K_W)^3 = -e + a + 3b \geq e + a - b + 4 = 2.$$

Since $-K_W$ is neither nef or big,

$$-K_W \cdot C_0 = a + b < 0.$$

The absurdity comes from

$$0 = e + a - b + 2 \leq a + b < 0.$$

Then the claim above yields

$$-e + 2b = 1. \tag{10}$$

From (9) and (10), $-1 = a + b$ and $a = -\frac{1}{2}e - \frac{3}{2}$. Then

$$-1 = -K_W \cdot C_0 = (-K_Y + E) \cdot C_1 \geq E \cdot C_1 = \mathcal{N}_{E/Y} \cdot C_1$$
$$= (-C_1 + aF) \cdot C_1 = e + a = \frac{1}{2}e - \frac{3}{2}.$$

Hence

$$e = 0, 1.$$

Therefore (10) induces $e = 1$. Hence E is a Hirzebruch surface of degree 1 over a smooth rational curve C_0. But this implies that $q(Y) = 0$ by Proposition 3.5 in [3].

Case 2: $g = 0$ and $c = 1$.

Now $e + a - b = -1$ by (8), and so by the claim above, we get

$$(-K_W)^3 = -e + a + 3b \geq e + a - b + 2 = 1.$$

Thus

$$-K_W \cdot C_0 = a + b < 0.$$

Then the absurdity comes from

$$-1 = e + a - b < a + b < 0.$$

Case 3: $g = 0$ and $c = 2$.
In this case, $e + a - b = 0$ by (8). We can derive contradiction in the similar way as Case 2.

Case 4: $g = 1$ and $c = 0$.
In this case, $e + a - b = 0$, and C_0 and E are smooth by the definition of c. If $-e + 2b > 0$, we obtain that $-K_W$ is nef and big easily. So we know $-e + 2b = 0$ and

$$0 \leq -K_Y|_E \cdot C_1 = -e + b = -\frac{1}{2}e.$$

Hence $e = 0$ or -1 because E is an elliptic ruled surface. Combining this with $e = 2b$, we know $e = b = 0$. Therefore $K_Y \cdot C_1 = 0$. If $\dim f(E) = 1$, then there exist infinitely many irreducible curves on E such that each curve is contracted by f. Take such an irreducible curve C on E. Then by (3), C is numerically proportional to C_1 in $N^1(E)$. If $\dim f^{-1}(f(C)) = 1$, then C is a smooth rational curve. But this derives contradiction because $\varphi(C)$ is an elliptic curve by $g = 1$. Thus there exists a prime divisor D such that $f(D) = f(C)$. This is also absurd because f never contracts infinitely many divisors. Therefore we get that $\dim f(E) = 2$. Let H be a general hyperplane section of $f(E)$. Then $H' := (f|_E)^*H$ is an irreducible curve on E which dose not intersect C_1. So we know that $H' \equiv dC_1$ for some $d > 0$ (cf. Chap.V, 2.20 in [8]). But since $f(H')$ is not a point, this lies in contradiction. $\square$

2 Proof of $\mathrm{alb}(X) \neq 2$

The proof of the following result is similar to the one of Theorem 1 in [27], which treats the smooth case in arbitrary dimension.

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Proposition 2.1. *Let X be a canonical 3-fold with nef anticanonical divisor and $\alpha : X \to \mathrm{Alb}(X)$ its Albanese morphism. Assume $\mathrm{alb}(X) = 2$. Then α is an algebraic fiber space.*

Proof. It is enough to consider the case that X is a terminal 3-fold by terminalization. Since a general hyperplane section of a terminal 3-fold is smooth, we can use the proof of [27]. □

Proposition 2.2. [16] *Let $\varphi : X \to Y$ be a surjective morphism between smooth varieties. Assume that the general fiber F is connected and $h^i(F, \mathcal{O}_F) = 0$ for $i > 0$. Then $h^i(X, \mathcal{O}_X) = h^i(Y, \mathcal{O}_Y)$ for all i.*

Proposition 2.3. [22] *Let X be a smooth projective 3-fold with nef $-K_X$ and $q(X) = 2$. We stand α for the Albanese morphism of X and $\varphi : Y \to X$ for a divisorial contraction from a terminal 3-fold Y. Assume that α is a smooth morphism. Then $-K_Y$ is not almost nef.*

Proposition 2.4. *Let X be a canonical 3-fold with strictly nef $-K_X$. Then $\mathrm{alb}(X) \neq 2$.*

Proof. Assume that $\mathrm{alb}(X) = 2$. Write $A := \mathrm{Alb}(X)$. Now that the Albanese morphism α is an algebraic fiber space (Proposition 2.1), a general fiber of α is $\mathbb{P}^1$. Using Proposition 2.2, we know $h^i(\mathcal{O}_X) = h^i(\mathcal{O}_A)$.

Let $f : Y \to X$ be a $\mathbb{Q}$-factorial terminalization as in the proof of Proposition 1.13. Since K_Y is not nef, there exists an extremal contraction $\varphi : Y \to W$ and we know $\dim W \geq 2$ because Proposition 2.1 says in paticular that α is surjective.

Case: $\underline{\dim W = 2}$

Now $h^0(K_W) = h^2(\mathcal{O}_W) = h^2(\mathcal{O}_X) = 1$, and so $K_W \cdot H \geq 0$ for some ample divisor H on Y. On the other hand, since $-(4K_W + \Delta)$ is almost nef (Proposition 1.8), $-(4K_W + \Delta) \cdot H \geq 0$. Thus $K_W \equiv 0$. So we know that W is an abelian surface and hence β is an isomorphism where $\beta : W \to \mathrm{Alb}(W)(= A)$ is the Albanese morphism.

If X is not $\mathbb{Q}$-factorial terminal, there exists a curve C on Y such that $f(C)$ is a point. On the other hand, $K_Y \cdot C = 0$ implies $\varphi(C)$ is a curve. This contradicts $\varphi = \alpha \circ f$. Hence X is a $\mathbb{Q}$-factorial terminal 3-fold and we can apply Proposition 1.7, so we know that X is smooth. Then $-K_X$ is ample by Theorem 0.3 (ii). The ampleness of $-K_X$ yields a cotradiction

with alb$(X) = 2$.

Case: $\dim W = 3$

The following argument is in pages 493-494 of [22]. By Mori Theory, there exists a sequence

$$Y \to W =: Y_1 \to \cdots \to Y_{n-1} \to Y_n \to Z.$$

where each morphism $\varphi_i : Y_i \to Y_{i+1}$ is a divisorial contraction or a flip for $i \geq 1$ and $\psi : Y_n \to Z$ is a Mori fiber space to a surface Z. As in the case $\dim W = 2$ above, we know that Z is an abelian surface, so we get readily that ψ is the Albanese morphism of Y_n. So by Proposition 1.7, ψ is a smooth morphism and $-K_{Y_n}$ is nef. Thus φ_{n-1} is divisorial. But this is absurd with Proposition 1.8 and Proposition 2.3. $\square$

3 Proof of the Main Theorem

We first prepare several lemmas in order to prove Theorem 3.8.

Lemma 3.1. *Let X be a canonical 3-fold with $-K_X$ nef and $\kappa(X) = -\infty$. Then $h^2(\mathcal{O}_X) < 2$.*

Proof. Taking a $\mathbb{Q}$-factorial terminalization, we may assume that X has at most $\mathbb{Q}$-factorial terminal singularities. By Mori theory, X is birational to a $\mathbb{Q}$-factorial terminal 3-fold W such that W has a Mori fiber space structure $\varphi : W \to Z$, that is, φ is an extremal contraction with $\dim W > \dim Z$. If $h^2(\mathcal{O}_X) \geq 2$, then Z is a projective surface and $h^0(K_Z) = h^2(\mathcal{O}_Z) = h^2(\mathcal{O}_X) \geq 2$. So $K_Z \cdot H \geq 1$ for some ample divisor H. On the other hand, since $-(4K_Z + \Delta) \cdot H \geq 0$ (Proposition 1.8), $-K_Z \cdot H \geq 0$. This is absurd. $\square$

Lemma 3.2. *Let X be a canonical 3-fold such that $-K_X$ is strictly nef. Assume $q(X) = 0$. Then $\pi_1^{alg}(X)$ is finite.*

Proof. Assume that $\pi_1^{alg}(X)$ is infinite. Then there exists an infinite tower of normal finite étale Galois covers

$$\cdots \to X_2 \to X_1 \to X_0 := X.$$

We know $\chi(\mathcal{O}_{X_m}) \gg 0$ by $\chi(\mathcal{O}_X) > 0$, hence $h^2(\mathcal{O}_{X_m}) \gg 0$ for a sufficiently large m. Then $-K_{X_m}$ being strictly nef contradicts Lemma 3.1. $\square$

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Lemma 3.3. [21] *Let S be a smooth projective surface with finite algebraic fundamental group $\pi_1^{alg}(S)$. Suppose we have a nef divisor D on S such that $D^2 = 0$. Then either $H^1(S, -D) = 0$, or there exists a positive integer n such that $H^0(S, nD) \neq 0$.*

The following lemma is the generalization of Lemma 3.8 in [25].

Lemma 3.4. *Let X be a canonical 3-fold with strictly nef $-K_X$. Let U stand for the locus of cDV points of X. Assume that $K_X{}^2 \equiv 0$ and $\pi_1^{alg}(U)$ is finite. Then $\chi(\mathcal{O}_X) \leq 0$.*

Proof. Because $\pi_1^{alg}(U) = \pi_1^{alg}(S)$ for a general hyperplane section S by Théorème 3.10, page 123 of [6], $\pi_1^{alg}(S)$ is finite. Assume that $\chi(\mathcal{O}_X) > 0$ below. Write $r := \operatorname{index}(X)$. Let $f : Y \to X$ be a terminalization and set $T := f^*S$. Since $-K_Y$ is nef and not big, $c_1(Y)c_2(Y) \geq 0$ by Proposition 1.1 and Remark 1.2. If $h^0(-rK_Y) = h^0(-rK_X) \neq 0$, then $-K_X$ is ample by Theorem 0.3 (iii) and so it contradicts $K_X{}^2 \equiv 0$. Hence Riemann-Roch Theorem (cf. [24]) says that

$$\chi(-rK_Y) = \frac{1}{12}(-rK_Y)c_2(Y) + \chi(\mathcal{O}_Y) > 0,$$

which implies $h^2(-rK_Y) > 0$. Let us consider the exact sequence

$$\cdots \to H^1(-rK_Y + T) \to H^1(-rK_Y + T|_T) \to H^2(-rK_Y) \to$$
$$\to H^2(-rK_Y + T) \to \cdots .$$

Exchange S for its multiple if necessary. Then we may assume $h^i(-rK_Y + T) = h^i(-rK_X + S) = 0$ for $i > 0$. Therefore

$$h^1((r+1)K_Y|_T) = h^1(-(r+1)K_Y|_T + K_T) = h^1(-rK_Y + T|_T) \neq 0.$$

Now that $\pi_1^{alg}(T) = \pi_1^{alg}(S)$ (cf. Lemma 3.1 in [26]) and T is smooth, we can apply Lemma 3.3 and we get $h^0(-nK_X|_S) = h^0(-nK_Y|_T) \neq 0$ for some $n > 0$. Hence

$$0 < -K_X \cdot (-nK_X|_S) = n(-K_X)^2 \cdot S = 0.$$

This is a contradiction. So $\chi(\mathcal{O}_X) \leq 0$. $\qquad\square$

Remark 3.5. Let X be a canonical 3-fold with strictly nef $-K_X$. Assume $(-K_X)^3 > 0$. Then we know that $-K_X$ is ample. In fact, the complete linear system $|-nK_X|$ is free for $n \gg 0$ by the Base Point Free Theorem (cf. [13]). Thus it yields a morphism $\Phi : X \to \mathbb{P}^N$. If there exists a curve C on X such that $\Phi(C)$ is a point, then $-nK_X \cdot C = \Phi^* H \cdot C = 0$ for a hyperplane H in $\mathbb{P}^N$, which contradicts the strictly nefness of $-K_X$. Therefore Φ is finite, and we obtain that $-K_X$ is ample.

Lemma 3.6. *Let X be a canonical 3-fold with strictly nef $-K_X$. Assume that $q(X) = 0$ and $K_X{}^2 \not\equiv 0$. Then $-K_X$ is ample.*

Proof. Let S be a sufficiently ample general hyperplane section and $r :=$ index(X). Consider the exact sequence

$$\longrightarrow \quad H^1(-rK_X + S|_S) \quad \longrightarrow \quad H^2(-rK_X) \longrightarrow H^2(-rK_X + S) \longrightarrow \; .$$

$$\|$$
$$H^1(-(r+1)K_X|_S + K_S) \qquad\qquad\qquad\qquad\qquad 0$$

$$\|$$
$$0$$

Therefore $h^2(-rK_X) = 0$. Let $f : Y \to X$ be a terminalization. If $(-K_X)^3 = (-K_Y)^3 = 0$, then $c_1(Y)c_2(Y) \geq 0$ by Proposition 1.1 and Remark 1.2. Furthermore by $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X) > 0$,

$$\chi(-rK_Y) = \frac{1}{12}(-rK_Y)c_2(Y) + \chi(\mathcal{O}_Y) > 0.$$

So $h^0(-rK_X) \neq 0$. This implies $-K_X$ is ample by Theorem 0.3 (iii). On the other hand, $-K_X$ is ample when $(-K_X)^3 > 0$ (Remark 3.5). This completes the proof. $\square$

Remark 3.7. Let X be a canonical 3-fold with strictly nef $-K_X$. Then we have $q(X) = 0$. In fact, because $\alpha(X)$ is a subvariety of Alb(X), $\kappa(\alpha(X)) \geq 0$. So alb$(X) = 3$ induces a contradiction with $\kappa(X) = -\infty$. Thus Proposition 1.13 and Proposition 2.4 deduce that alb$(X) = 0$, which is equivalent to $q(X) = 0$.

We are now in the position to prove the main theorem of this paper. The proof is based on the Miyaoka's idea in [21].

Theorem 3.8. *Let X be a canonical 3-fold with strictly nef $-K_X$. Then $-K_X$ is ample.*

Proof. Note that $q(X) = 0$ by Remark 3.7. Let U be the locus of cDV points of X. If $\pi_1^{alg}(U)$ is finite, combining Lemma 3.4 and Lemma 3.6, we know that $-K_X$ is ample because $\chi(\mathcal{O}_X) > 0$.

If $\pi_1^{alg}(U)$ is infinite, then there exists an infinite tower of normal finite Galois covers

$$\cdots \to X_2 \to X_1 \to X_0 := X$$

such that $f_n|_{U_n} : U_n \to U$ is étale where f_n is the morphism from X_n to X and $U_n := f_n^{-1}(U)$. Let x be a non-cDV point in X and g_n the morphism from X_n to X_{n-1}. Since $\pi_1^{alg}(V^o)$ is a finite group (Theorem 3.6 in [26]) where V is a sufficiently small analytic neighbourhood of x and V^o stands for the smooth locus of V (we use ^{o} below as the same way), there exists a positive integer m_x such that $f_{n+1}^{-1}(V)^o$ is disjoint union of copies of $f_n^{-1}(V)^o$ for all $n \geq m_x$. Now that X_n is normal, for all $y \in f_n^{-1}(x)$, $f_{n+1}^{-1}(V)^o \cup g_{n+1}^{-1}(y)$ is also disjoint union of copies of $f_n^{-1}(V)^o \cup y$ (see Théorème 3.6, page 38 of [6]) and so g_n is étale over y. X has only finitely many non-cDV points, so g_n is étale and $\pi_1^{alg}(X_n)$ is infinite for all sufficiently large n. Now that X_n is a canonical 3-fold with strictly nef $-K_{X!_n}$, we derive a contradiction from Lemma 3.2. □

From Lemma 3.1, we also obtain the following result:

Theorem 3.9. (cf. [27]) *Let X be a canonical 3-fold with nef $-K_X$. Then $q(X) \leq 3$. Furthermore $q(X) = 3$ if and only if X is an abelian variety.*

Proof. Assume that $\kappa(X) \geq 0$. Then we know readily that $\kappa(X) = 0$. Let $f : Y \to X$ be a resolution of singularities. Because the Albanese morphism of Y is an algebraic fiber space ([9]), so is the one of X, $\alpha : X \to \mathrm{Alb}(X) =: A$ and $q(X) \leq 3$. Consider the case that the equality holds. Then we can regard terminalization of X as a resolution of singularities f (Corollary 8.4 in [10]). So Y is an abelian variety by [27]. Hence f and α are isomorphic.

If $\kappa(X) = -\infty$, it is enough to consider the case that $-K_X$ is not big. According to Proposition 1.1 and Remark 1.2, $\chi(\mathcal{O}_X) \geq 0$ so we get $q(X) < 3$ by Lemma 3.1. □

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